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The asymptotic number of weighted partitions with a given number of parts

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Abstract For a given sequence b_k of non-negative real numbers, the number of weighted partitions of a positive integer n having m parts $c_{n,m}$ has bivariate generating function equal to $\prod_{k=1}^{\infty}(1-yz^k)^{-b_k}$. Under the assumption that $b_k \sim Ck^{r-1}$, $r > 0$, and related conditions on the Dirichlet generating function of the weights b_k, we find asymptotics for $c_{n,m}$ when $m = m(n)$ satisfies $m = o\left(n^{\frac{r}{r+1}}\right)$ and $\lim_{n\to\infty} m/\log^{3+\epsilon} n = \infty$, $\epsilon > 0$.

Keywords weighted partition · asymptotic analysis · Meinardus' method · bivariate generating function

Mathematics Subject Classification (2010) 11P82

1 Introduction and Background

This study is devoted to the asymptotic formula for the quantity $c_{n,m}$ which denotes the number of weighted integer partitions of n, having exactly $1 \le m \le n$ parts. The weights are a sequence of real numbers b_k, $k \ge 1$ and the ordinary bivariate generating function $f(y,z)$ for the sequence $c_{n,m}$ is

$$f(y,z) = \sum_{n=1}^{\infty}\sum_{m=1}^{n} c_{n,m} y^m z^n = \prod_{k=1}^{\infty}(1-yz^k)^{-b_k}. \tag{1}$$

Let

$$D(s) = \sum_{k=1}^{\infty} \frac{b_k}{k^s}$$

be the Dirichlet generating function for the sequence of weights. We make the following assumptions on $D(s)$:

(i) Let $s = \sigma + it$. For constants $r > 0$ and $1 < C_0 < 2$ the Dirichlet series $D(s)$ converges in the half-plane $\sigma > r > 0$ and the function $D(s)$ has an analytic continuation to the half-plane

$$\mathcal{H} = \{s : \sigma \ge -C_0\} \tag{2}$$

on which it is analytic except for a simple pole at $s = r$ with residue $A > 0$.

(ii) There is a constant $C_1 > 0$ such that

$$D(s) = O\left(|t|^{C_1}\right), \quad t \to \infty \tag{3}$$

uniformly in $s \in \mathcal{H}$.

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(iii) There is a constant $C > 0$ such that

$$b_k \sim Ck^{r-1}. \tag{4}$$

The first two conditions are similar to assumptions of Meinardus [1], although we have assumed $1 < C_0 < 2$ in the second condition rather than the slightly weaker assumption of Meinardus [1] that $0 < C_0 < 1$. Meinardus' third condition did not make any direct assumptions on the b_k. He assumed

(iii)′ There are constants $C_2 > 0$ and $\nu > 0$, such that the function $g(x) = \sum_{k=1}^{\infty} b_k e^{-kx}$, $x = \delta + 2\pi i\alpha$, α real and $\delta > 0$ satisfies

$$\Re(g(x)) - g(\delta) \leq -C_2\delta^{-\nu}, \quad |\arg(x)| > \pi/4,\ 0 \neq |\alpha| \leq 1.2,$$

for small enough values of δ.

Meinardus [1] introduced his conditions in an analysis of $c_n = \sum_{m=1}^{n} c_{n,m}$ with generating function $f(1, z)$. Granovsky et al. [7] weakened condition (iii)′ and obtained the asymptotics of c_n under

(iii)″ For small enough $\delta > 0$ and any $\mu > 0$,

$$\sum_{k=1}^{\infty} b_k e^{-k\delta} \sin^2(\pi k\alpha) \geq \left(1 + \frac{r}{2} + \mu\right) \frac{2}{\log 5} |\log \delta|$$

where $\sqrt{\delta} \leq \alpha \leq 1/2$.

Let ξ be a random variable having distribution

$$\mathbb{P}(\xi_n = m) = \frac{c_{n,m}}{c_n}, \quad 1 \leq m \leq n.$$

Haselgrove and Tempereley [2] obtained an expression for $c_{n,m}$ under several conditions, one of which implies $r < 2$ and conjectured that ξ_n should have a limiting Gaussian distribution for $r > 2$. Of particular interest is the case $b_k = k$ for which c_n is the number of plane partitions of n and ξ_n is the number of the sum of the diagonal parts; see [3]. Under conditions (i) (with $0 < C_0 < 1$), (ii), and (iii)″, Mutafchiev [4] found the limiting distribution of ξ_n for all $r > 0$. The non-Gaussian distributions for $r < 2$ had been discovered previously, as is explained in [4]. The Gaussian distributions for $r \geq 2$ confirmed the conjecture of [2].

Hwang [5] studied the number of components in a randomly chosen selection, partitions having no repeated parts, assuming Meinardus-type conditions and an analysis of a bivariate generating function analogous to (1).

In this paper we will find asymptotics of $c_{n,m}$ through an analysis of the bivariate function (1) which adapts the methods used in Granovsky et. al. [6–9] for finding the asymptotics of the coefficients of univariate functions including $f(1, z)$. The initiator of the method was Meinardus [1]. Our main result is stated in terms of functions of n and m defined in (8) and (9). Let

$$\mathrm{Li}_s(z) = \sum_{j=1}^{\infty} \frac{z^j}{j^s} \tag{5}$$

be the polylogarithmic function of order s and define

$$\Lambda(\mu) = \mathrm{Li}_{r+1}(e^{-\mu})^{-r} \mathrm{Li}_r(e^{-\mu})^{r+1}.$$

The asymptotic

$$\mathrm{Li}_s(e^{-\mu}) = e^{-\mu} + O(e^{-2\mu}), \quad \mu \to \infty, \tag{6}$$

which holds uniformly for all $s \in \mathcal{H}$ results in $\Lambda(\mu) \sim e^{-\mu}$. The identity which holds for all s

$$\frac{\partial \mathrm{Li}_s(e^{-\mu})}{\partial \mu} = -\mathrm{Li}_{s-1}(e^{-\mu}), \tag{7}$$

implies that

$$\begin{aligned}\frac{d}{d\mu}\Lambda(\mu) &= r\mathrm{Li}_{r+1}(e^{-\mu})^{-r-1}\mathrm{Li}_r(e^{-\mu})^{r+2} - (r+1)\mathrm{Li}_{r+1}(e^{-\mu})^{-r}\mathrm{Li}_r(e^{-\mu})^r\mathrm{Li}_{r-1}(e^{-\mu}) \\ &= -e^{-\mu} + O(e^{-2\mu}), \quad \mu \to \infty,\end{aligned}$$

where the implicit constant in the $O(\cdot)$ term depends on r. Therefore, for some $\mu_0 > 0$, $\Lambda(\mu)$ decreases monotonically to 0 for when restricted to $\mu > \mu_0$. Taking now Λ restricted to (μ_0, ∞), it follows that Λ has an inverse Λ^{-1}. Assuming that $m = o\left(n^{\frac{r}{r+1}}\right)$ and letting $h_r = A\Gamma(r)$, for n large enough define

$$\mu_{n,m} = \Lambda^{-1}\left(\frac{r^r m^{r+1} n^{-r}}{h_r}\right) \tag{8}$$

and

$$\delta_{m,n} = \frac{rm\mathrm{Li}_{r+1}(e^{-\mu_{m,n}})}{n\mathrm{Li}_r(e^{-\mu_{m,n}})}. \tag{9}$$

Note that $\mu_{n,m} \to \infty$ as $n \to \infty$ and

$$e^{-\mu_{n,m}} \sim \Lambda(\mu_{n,m}) \sim r^r m^{r+1} n^{-r}/h_r \tag{10}$$

and

$$\delta_{n,m} \sim rm/n \tag{11}$$

as $n \to \infty$.

Theorem 1 *Assume conditions (i), (ii) and (iii) above hold. If $m = m(n)$ is such that*

$$m = o\left(n^{\frac{r}{r+1}}\right), \tag{12}$$

and

$$\lim_{n\to\infty} m(n)/\log^{3+\epsilon} n = \infty \tag{13}$$

for some $\epsilon > 0$ then,

$$c_{n,m} \sim \exp\left((m+1)\mu_{n,m} + n\delta_{n,m} + h_r\delta_{n,m}^{-r}Li_{r+1}(e^{-\mu_{n,m}})\right)\frac{\delta_{n,m}^{r+1}}{2\pi C\sqrt{r\Gamma(r)}} \tag{14}$$

where

$$h_r = A\Gamma(r). \tag{15}$$

If

$$m = o\left(n^{\frac{r}{r+2}}\right), \tag{16}$$

then if we set

$$\mu_{n,m} = -\log\left(\frac{r^r m^{r+1} n^{-r}}{h_r}\right) \tag{17}$$

and

$$\delta_{n,m} = \frac{rm}{n}, \tag{18}$$

using (6) in (35) and (37) produces (36) and (38). It follows that under (16) we may use (17) and (18) in Theorem 1 instead of (8) and (9).

If $r > 2$, then Theorem 1 of [4] implies that there is a constant $\kappa > 0$ such that $\mathbb{P}\left(\log^{3+\epsilon} n \le \xi_n \le \kappa n^{\frac{r-1}{r+1}}\right) = 1 - o(1)$ and so Theorem 1 covers the significant m with respect to the distribution of ξ_n. However, if $r \le 2$, then $\mathbb{P}(\xi_n \le m) = o(1)$ for any m satisfying (12).

The assumption (4) can probably be weakened to, say, $b_k \asymp k^{r-1}$ and an approximation to $c_{n,m}$ still obtained, but doing so with the methods of this paper would require at least the derivation or imposition of a lower bound on the left hand side of (45).

2 A fundamental identity

We will establish an expression for $c_{n,m}$ which is fundamental for our analysis of $c_{n,m}$. Define a truncation of $f(y,z)$ by

$$f_n(y,z) = \prod_{k=1}^{n} (1 - yz^k)^{-b_k} \tag{19}$$

Let X_k have p.d.f.

$$\mathbb{P}(X_k = l) = \binom{b_k + l - 1}{l} (1 - e^{-\mu - \delta k}) e^{-\mu l - \delta k l}, \quad l \geq 0,$$

a negative binomial distribution with parameters b_k and $e^{-\mu-\delta k}$, where the parameters $\mu > 0$, $\delta > 0$ are arbitrary, and let

$$Y_n = \sum_{k=1}^{n} X_k, \quad Z_n = \sum_{k=1}^{n} k X_k.$$

Lemma 1 *For any $\mu > 0$ and $\delta > 0$ we have*

$$c_{n,m} = e^{m\mu + n\delta} f_n(e^{-\mu}, e^{-\delta})\, \mathbb{P}(Y_n = m, Z_n = n).$$

Proof. Observe that

$$\begin{aligned} c_{n,m} &= \frac{1}{(2\pi i)^2} \oint \oint \frac{f(y,z)}{y^{m+1} z^{n+1}} dy\, dz \\ &= e^{m\mu + n\delta} \int_{-1/2}^{1/2} \int_{-1/2}^{1/2} f(e^{-\mu + 2\pi i \beta}, e^{-\delta + 2\pi i \alpha}) e^{-2\pi i \alpha n} e^{-2\pi i \beta m} d\alpha\, d\beta. \end{aligned}$$

It follows that

$$c_{n,m} = e^{m\mu + n\delta} \int_{-1/2}^{1/2} \int_{-1/2}^{1/2} f_n(e^{-\mu + 2\pi i \beta}, e^{-\delta + 2\pi i \alpha}) e^{-2\pi i \alpha n} e^{-2\pi i \beta m} d\alpha\, d\beta. \tag{20}$$

For $|\alpha| \leq 1/2$ and $|\beta| \leq 1/2$ we have

$$\begin{aligned} \mathbb{E}(e^{(2\pi i \alpha + 2\pi i \beta k) X_k}) &= \left(\frac{1 - e^{-\mu - \delta k}}{1 - e^{-\mu - \delta k + 2\pi i \alpha + 2\pi i \beta k}} \right)^{b_k} \\ &= \left(\frac{1 - e^{-\mu - \delta k}}{1 - e^{-\mu + 2\pi i \alpha - \delta k + 2\pi i \beta k}} \right)^{b_k}. \end{aligned}$$

Therefore, the joint characteristic function of Z_n and Y_n is

$$\begin{aligned} \phi_n(\alpha, \beta) &:= \mathbb{E}(e^{2\pi i (\alpha Y_n + \beta Z_n)}) \\ &= \prod_{k=1}^{n} \mathbb{E}(e^{2\pi i (\alpha + \beta k) X_k}) \\ &= \prod_{k=1}^{n} \left(\frac{1 - e^{-\mu - \delta k}}{1 - e^{-\mu + 2\pi i \alpha - \delta k + 2\pi i \beta k}} \right)^{b_k} \\ &= \frac{f_n(e^{-\mu + 2\pi i \alpha}, e^{-\delta + 2\pi i \beta})}{f_n(e^{-\mu}, e^{-\delta})}. \end{aligned} \tag{21}$$

We now combine (20) and (21) to obtain

$$\begin{aligned} c_{n,m} &= e^{m\mu + n\delta} f_n(e^{-\mu}, e^{-\delta}) \int_{-1/2}^{1/2} \int_{-1/2}^{1/2} \phi_n(\alpha, \beta) e^{-2\pi i \alpha m} e^{-2\pi i \beta n} d\alpha\, d\beta \\ &= e^{m\mu + n\delta} f_n(e^{-\mu}, e^{-\delta}) \mathbb{P}(Y_n = m, Z_n = n). \end{aligned}$$

■

In proving Theorem 1 we take $\mu = \mu_{n,m}$ and $\delta = \delta_{n,m}$ given by (8) and (9), giving

$$c_{n,m} = e^{m\mu_{n,m} + n\delta_{n,m}} f_n(e^{-\mu_{n,m}}, e^{-\delta_{n,m}})\, \mathbb{P}(Y_n = m, Z_n = n). \tag{22}$$

In Section 3 we estimate $f_n(e^{-\mu_{n,m}}, e^{-\delta_{n,m}})$ and in Section 4 we estimate $\mathbb{P}(Y_n = m, Z_n = n)$.

3 Asymptotics for the truncated generating function.

We first find the asymptotics of $f(e^{-\mu}, e^{-\delta})$.

Lemma 2 *We have*

$$\log f(e^{-\mu}, e^{-\delta}) = h_r \delta^{-r} Li_{r+1}(e^{-\mu}) + h_0 Li_1(e^{-\mu}) - h_{-1}\delta \mathrm{Li}_0(e^{-\mu}) + \Delta(\mu,\delta), \quad \mu,\delta > 0, \tag{23}$$

where h_r is given by (15), $h_0 = D(0)$, $h_{-1} = D(-1)$, and

$$\Delta(\mu,\delta) = \frac{1}{2\pi i}\int_{-C_0-i\infty}^{-C_0+i\infty} \delta^{-s}\Gamma(s)D(s)\mathrm{Li}_{s+1}(e^{-\mu})ds = O(\delta^{C_0}e^{-\mu}).$$

Proof. Substituting the expression of $e^{-\delta}$ as the inverse Mellin transform of the Gamma function:

$$e^{-\delta} = \frac{1}{2\pi i}\int_{v-i\infty}^{v+i\infty} \delta^{-s}\Gamma(s)\,ds, \quad \delta > 0, v > 0.$$

with v taken to be $v = 1 + r$ in (19) we obtain

$$\begin{aligned}
\log f(e^{-\mu}, e^{-\delta}) &= -\sum_{k=1}^{\infty} b_k \log(1 - e^{-\mu}e^{-\delta k}) \\
&= \sum_{k=1}^{\infty} b_k \sum_{j=1}^{\infty} \frac{e^{-\mu j}e^{-\delta k j}}{j} \\
&= \sum_{k=1}^{\infty} b_k \sum_{j=1}^{\infty} \frac{e^{-\mu j}}{j}\frac{1}{2\pi i}\int_{1+r-i\infty}^{1+r+i\infty} (\delta k j)^{-s}\Gamma(s)\,ds \\
&= \frac{1}{2\pi i}\int_{1+r-i\infty}^{1+r+i\infty} \delta^{-s}\Gamma(s)\sum_{k=1}^{\infty}\frac{b_k}{k^s}\sum_{j=1}^{\infty}\frac{e^{-\mu j}}{j^{s+1}}\,ds \\
&= \frac{1}{2\pi i}\int_{1+r-i\infty}^{1+r+i\infty} \delta^{-s}\Gamma(s)D(s)\mathrm{Li}_{s+1}(e^{-\mu})ds.
\end{aligned} \tag{24}$$

The function $\mathrm{Li}_{s+1}(e^{-\mu})$ defined (5) is analytic for all complex s for each $\mu > 0$ while by condition (i) the function $D(s)$ is assumed to be holomorhic in $\mathcal{H}$, with a unique simple positive pole r with a positive residue A. The gamma function has simple poles at $s = 0$ and $s = -1$ with residues 1 and -1 respectively. We will shift the countour of integration in (24) from $\{s : \Re(s) = 1 + r\}$ to $\{s : \Re(s) = -C_0\}$. In performing this shift we use (3), the fact that

$$|\mathrm{Li}_{s+1}(e^{-\mu})| \le \sum_{j=1}^{\infty}\frac{e^{-\mu j}}{j^{1-C_0}} = O(e^{-\mu}), \quad s \in \mathcal{H}, \tag{25}$$

and the bound

$$\Gamma(s) = O\left(\exp\left(-\frac{\pi}{2}|t|\right)|t|^{C_2}\right)$$

for a constant $C_2 > 0$; see [3]. The Cauchy residue theorem produces (23). ∎

We now are able to find the asymptotics of the second factor of (22).

Lemma 3 *Under the assumptions of Theorem 1,*

$$f_n(e^{-\mu_{n,m}}, e^{-\delta_{n,m}}) \sim \exp\left(h_r\delta_{n,m}^{-r}\mathrm{Li}_{r+1}(e^{-\mu_{n,m}})\right).$$

Proof. By Lemma 2, we have

$$\log f(e^{-\mu_{n,m}}, e^{-\delta_{n,m}}) = h_r\delta_{n,m}^{-r}\mathrm{Li}_{r+1}(e^{-\mu_{n,m}}) + o(1)$$

and so, by (1) and (19),

$$\log f_n(e^{-\mu_{n,m}}, e^{-\delta_{n,m}}) = h_r \delta_{n,m}^{-r} \mathrm{Li}_{r+1}(e^{-\mu_{n,m}}) + \sum_{k=n+1}^{\infty} b_k \log(1 - e^{-\mu_{n,m}} e^{-\delta_{n,m} k}) + o(1).$$

We estimate

$$\begin{aligned}
\sum_{k=n+1}^{\infty} b_k \log(1 - e^{-\mu_{n,m}} e^{-\delta_{n,m} k}) &= O\left(\sum_{k=n+1}^{\infty} b_k e^{-\mu_{n,m}} e^{-\delta_{n,m} k} \right) \\
&= O\left(e^{-\mu_{n,m}} \sum_{k=n+1}^{\infty} k^{r-1} e^{-\delta_{n,m} k} \right) \\
&= O\left(e^{-\mu_{n,m}} \delta_{n,m,}^{-r} \int_{n\delta_{n,m}}^{\infty} t^{r-1} e^{-t}\, dt \right) \\
&= o\left(e^{-\mu_{n,m}} \delta_{n,m,}^{-r} \right),
\end{aligned}$$

where we have used $n\delta_{n,m} \to \infty$ which follows from (11) and (13). ■

4 The local limit theorem

We have found asymptotics for the first two factors of (22) and we now will them for the third factor. The proof of the following Local Limit Lemma is similar in places to one in [6].

Lemma 4 (Local Limit Lemma) *Under the assumptions of Theorem 1,*

$$\mathbb{P}(Y_n = m, Z_n = n) \sim \frac{e^{\mu_{n,m}} \delta_{n,m}^{r+1}}{2\pi C \sqrt{r \Gamma(r)}} \tag{26}$$

Proof. Define

$$\alpha_0(n) = e^{\mu_{n,m}/2} \delta_{n,m}^{r/2} \log^{(4+\epsilon)/8} n \tag{27}$$

and

$$\beta_0(n) = e^{\mu_{n,m}/2} \delta_{n,m}^{r/2+1} \log^{(8+\epsilon)/16} n \tag{28}$$

The asymptotics (10) and (11) imply

$$e^{\mu_{n,m}/2} \delta_{n,m}^{r/2} \asymp (m^{r+1} n^{-r})^{-1/2} (mn^{-1})^{r/2} = m^{-1/2} = o(\log^{-(3+\epsilon)/2} n)$$

by (13). Therefore $\alpha_0(n) = o(1)$ and similarly $\beta_0(n) = o(1)$. Let

$$R_n = [-\alpha_0(n), \alpha_0(n)] \times [-\beta_0(n), \beta_0(n)]$$

and

$$\overline{R_n} = ([-1/2, 1/2] \times [-1/2, 1/2]) \setminus R_n.$$

We express $\mathbb{P}(Y_n = m, Z_n = n)$ in (22) as

$$\mathbb{P}(Y_n = m, Z_n = n) = I_1 + I_2, \tag{29}$$

where

$$I_1 = \int\int_R \phi_n(\alpha, \beta) e^{-2\pi i(\alpha n + \beta m)} d\alpha d\beta \tag{30}$$

and

$$I_2 = \int\int_{\overline{R}} \phi_n(\alpha, \beta) e^{-2\pi i(\alpha n + \beta m)} d\alpha d\beta. \tag{31}$$

We will estimate I_1 and I_2 separately.

Estimate of I_1

Expanding $\log\phi_n(\alpha,\beta)$ into a Taylor series centred at $\alpha_0=0$, $\beta_0=0$ for $(\alpha,\beta)\in R_n$ gives

$$\log\phi_n(\alpha,\beta)=2\pi i\alpha(\mathbb{E}Y_n)+2\pi i\beta(\mathbb{E}Z_n)-2(\pi\alpha)^2\mathrm{Var}(Y_n)-2(\pi\beta)^2\mathrm{Var}(Z_n)$$
$$-(2\pi)^2\alpha\beta\mathrm{Cov}(Y_n,Z_n)+O\left(\max_{0\le s\le 3}|\rho_s|\alpha_0^s\beta_0^{3-s}\right), \tag{32}$$

where

$$\rho_s=\frac{\partial^3}{\partial\alpha^s\partial\beta^{3-s}}\log\phi_n(\alpha,\beta)\Big|_{\alpha=0,\beta=0}.$$

It follows from (21) that

$$\mathbb{E}(Y_n)=\frac{1}{2\pi i}\frac{\partial}{\partial\alpha}\log\phi_n(\alpha,0)\Big|_{\alpha=0}=-\frac{\partial}{\partial\mu}\log f_n(e^{-\mu},e^{-\delta_{n,m}})\Big|_{\mu=\mu_{n,m}}$$

and

$$\mathbb{E}(Z_n)=\frac{1}{2\pi i}\frac{\partial}{\partial\beta}\log\phi_n(0,\beta)\Big|_{\beta=0}=-\frac{\partial}{\partial\delta}\log f_n(e^{-\mu_{n,m}},e^{-\delta})\Big|_{\delta=\delta_{n,m}}$$

Therefore, (1), (19), (23) and an estimate similar to one in the proof of Lemma 3 imply

$$\mathbb{E}(Y_n)=-\frac{\partial}{\partial\mu}\log f(e^{-\mu},e^{-\delta_{n,m}})\Big|_{\mu=\mu_{n,m}}-\sum_{k=n+1}^{\infty}b_k\frac{e^{-\mu_{n,m}-k\delta_{n,m}}}{1-e^{-\mu_{n,m}-k\delta_{n,m}}}$$
$$=h_r\delta_{n,m}^{-r}Li_r(e^{-\mu_{n,m}})+h_0Li_0(e^{-\mu_{n,m}})-h_{-1}\delta_{n,m}\mathrm{Li}_{-1}(e^{-\mu_{n,m}})$$
$$-\frac{\partial}{\partial\mu}\Delta(\mu,\delta_{n,m})\Big|_{\mu=\mu_{n,m}}+O\left(e^{-\mu_{n,m}}\delta_{n,m,}^{-r}\int_{n\delta_{n,m}}^{\infty}t^{r-1}e^{-t}\,dt\right) \tag{33}$$

and similarly

$$\mathbb{E}(Z_n)=-\frac{\partial}{\partial\delta}\log f(e^{-\mu_{n,m}},e^{-\delta})\Big|_{\delta=\delta_{n,m}}-\sum_{k=n+1}^{\infty}kb_k\frac{e^{-\mu_{n,m}-k\delta_{n,m}}}{1-e^{-\mu_{n,m}-k\delta_{n,m}}}$$
$$=h_r r\delta_{n,m}^{-r-1}Li_{r+1}(e^{-\mu_{n,m}})+h_{-1}\mathrm{Li}_0(e^{-\mu_{n,m}})$$
$$-\frac{\partial}{\partial\delta}\Delta(\mu_{n,m},\delta)\Big|_{\delta=\delta_{n,m}}+O\left(e^{-\mu_{n,m}}\delta_{n,m,}^{-r-1}\int_{n\delta_{n,m}}^{\infty}t^{r}e^{-t}\,dt\right) \tag{34}$$

where we have used (7). By using the method of the proof of Lemma 2 of [7] and (25) we obtain

$$\frac{\partial}{\partial\mu}\Delta(\mu,\delta_{n,m})\Big|_{\mu=\mu_{n,m}}=O\left(e^{-\mu_{n,m}}\delta_{n,m}^{C_0}\right)=o(1)$$

and

$$\frac{\partial}{\partial\delta}\Delta(\mu_{n,m},\delta)\Big|_{\delta=\delta_{n,m}}=O\left(e^{-\mu_{n,m}}\delta_{n,m}^{C_0-1}\right)=o(1),$$

where we used the assumption $C_0>1$ in the last step. Moreover,

$$\int_{n\delta_{n,m}}^{\infty}t^{r-1}e^{-t}\,dt\le\int_{n\delta_{n,m}}^{\infty}e^{-t/2}\,dt=2e^{-n\delta_{n,m}/2},$$

where the inequality holds for n large enough, and consequently (10), (11) and (13) show that the $O(\cdot)$ terms in (33) and (34) are of order $o(1)$. It follows from (8) and (9) that

$$\mathbb{E}(Y_n)=h_r\delta_{n,m}^{-r}Li_r(e^{-\mu_{n,m}})+o(1) \tag{35}$$
$$=h_r\left(\frac{rm}{n}\right)^{-r}\mathrm{Li}_{r+1}(e^{-\mu_{n,m}})^{-r}\mathrm{Li}_r^{r+1}(e^{-\mu_{n,m}})+o(1)$$
$$=h_r\left(\frac{rm}{n}\right)^{-r}\Lambda(\mu_{n,m})+o(1)$$
$$=h_r\left(\frac{rm}{n}\right)^{-r}\left(\frac{r^rm^{r+1}n^{-r}}{h_r}\right)+o(1)$$
$$=m+o(1) \tag{36}$$

and

$$\begin{aligned}
\mathbb{E}(Z_n) &= h_r r \delta_{n,m}^{-r-1} Li_{r+1}(e^{-\mu_{n,m}}) + o(1) &(37)\\
&= h_r r \left(\frac{rm}{n}\right)^{-r-1} \mathrm{Li}_{r+1}(e^{-\mu_{n,m}})^{-r} \mathrm{Li}_r^{r+1}(e^{-\mu_{n,m}}) + o(1)\\
&= h_r r \left(\frac{rm}{n}\right)^{-r-1} \Lambda(\mu_{n,m}) + o(1)\\
&= h_r r \left(\frac{rm}{n}\right)^{-r-1} \left(\frac{r^r m^{r+1} n^{-r}}{h_r}\right) + o(1)\\
&= n + o(1). &(38)
\end{aligned}$$

We also have to estimate the $|\rho_s|$. We have

$$\begin{aligned}
\rho_s &= \frac{\partial^3}{\partial\alpha^s \partial\beta^{3-s}} \left(-\sum_{k=1}^n b_k \log\left(1 - e^{-\mu_{n,m}+2\pi i\alpha - \delta_{n,m}k + 2\pi i\beta k}\right)\right)\Big|_{\alpha=0,\beta=0}\\
&= \frac{\partial^3}{\partial\alpha^s \partial\beta^{3-s}} \left(\sum_{k=1}^n b_k \sum_{j=1}^\infty \frac{1}{j} e^{j(-\mu_{n,m}+2\pi i\alpha - \delta_{n,m}k + 2\pi i\beta k)}\right)\Big|_{\alpha=0,\beta=0}\\
&= \sum_{k=1}^n b_k \sum_{j=1}^\infty \frac{1}{j} e^{-j\mu_{n,m} - j\delta_{n,m}k} (2\pi i j)^s (2\pi i k j)^{3-s}
\end{aligned}$$

so

$$\begin{aligned}
|\rho_s| &\le \sum_{k=1}^n b_k k^{3-s} e^{-\delta_{n,m}k} \sum_{j=1}^\infty \frac{1}{j} e^{-j\mu_{n,m}} (2\pi j)^3\\
&= O\left(e^{-\mu_{n,m}} \sum_{k=1}^n k^{r-s+2} e^{-\delta_{n,m}k}\right)\\
&= O\left(e^{-\mu_{n,m}} \delta_{n,m}^{-r+s-3}\right).
\end{aligned}$$

Use of (10), (11), (27), and (28) shows that for $0 \le s \le 3$,

$$\begin{aligned}
|\rho_s| \alpha_0^s \beta_0^{3-s} &= O(e^{\mu_{n,m}/2} \delta_{n,m}^{r/2} \log^{(24+2\epsilon s + (3-s)\epsilon)/16} n)\\
&= O\left(\left(m^{r+1} n^{-r}\right)^{-1/2} \left(mn^{-1}\right)^{r/2} \log^{(12+3\epsilon)/8} n\right)\\
&= O(m^{-1/2} \log^{(12+3\epsilon)/8} n)
\end{aligned}$$

and so (13) results in

$$\max_{0\le s\le 3} |\rho_s| \alpha_0^s \beta_0^{3-s} = o(n^{-\epsilon/8}). \tag{39}$$

It now follows from (30), (32), (36), (38), and (39) that

$$I_1 \sim \int\int_{R_n} \exp\left(-2\pi^2 \left\{\mathrm{Var}(Y_n)\alpha^2 + 2\mathrm{Cov}(Y_n, Z_n)\alpha\beta + \mathrm{Var}(Z_n)\beta^2\right\}\right) d\alpha d\beta.$$

Let us define the matrix Σ_n by

$$\Sigma_n = \begin{pmatrix} \mathrm{Var}(Y_n) & \mathrm{Cov}(Y_n, Z_n) \\ \mathrm{Cov}(Y_n, Z_n) & \mathrm{Var}(Z_n) \end{pmatrix}$$

so that

$$I_1 \sim \int_{R_n} \exp\left(-2\pi^2 \begin{pmatrix}\alpha\\ \beta\end{pmatrix}^{\mathrm{T}} \Sigma_n \begin{pmatrix}\alpha\\ \beta\end{pmatrix}\right) d\alpha d\beta,$$

where T denotes transpose. Since Σ_n is positive definite and symmetric it has a square root $\sqrt{\Sigma_n}$. Define the variables u and v by

$$\begin{pmatrix} u \\ v \end{pmatrix} = \sqrt{\Sigma_n} \begin{pmatrix} \alpha \\ \beta \end{pmatrix}.$$

Change of variables gives

$$\begin{aligned} I_1 &\sim \int\int_{S_n} \frac{1}{|\det(\sqrt{\Sigma_n})|} e^{-2\pi^2(u^2+v^2)} du\, dv \\ &= \frac{1}{\sqrt{\mathrm{Var}(Z_n)\mathrm{Var}(Y_n) - \mathrm{Cov}(Z_n,Y_n)^2}} \int\int_{S_n} e^{-2\pi^2(u^2+v^2)} du\, dv. \end{aligned}$$

where S_n is the image of R_n under the map $\sqrt{\Sigma_n}$. Under the assumption (iii) that $b_k \sim Ck^{r-1}$, and using $n\delta_{n,m} \to \infty$, we have

$$\mathrm{Var}(Y_n) = \sum_{k=1}^{n} b_k \frac{e^{-\mu_{n,m}} e^{-\delta_{n,m}k}}{1 - e^{-\mu_{n,m}} e^{-\delta_{n,m}k}} \sim e^{-\mu_{n,m}} \sum_{k=1}^{n} Ck^{r-1} e^{-\delta_{n,m}k} \sim C\Gamma(r) e^{-\mu_{n,m}} \delta_{n,m}^{-r}, \tag{40}$$

$$\mathrm{Cov}(Y_n, Z_n) = \sum_{k=1}^{n} kb_k \frac{e^{-\mu_{n,m}} e^{-\delta_{n,m}k}}{1 - e^{-\mu_{n,m}} e^{-\delta_{n,m}k}} \sim e^{-\mu_{n,m}} \sum_{k=1}^{n} Ck^{r} e^{-\delta_{n,m}k} \sim C\Gamma(r+1) e^{-\mu_{n,m}} \delta_{n,m}^{-r-1}, \tag{41}$$

and

$$\mathrm{Var}(Z_n) = \sum_{k=1}^{n} k^2 b_k \frac{e^{-\mu_{n,m}} e^{-\delta_{n,m}k}}{1 + e^{-\mu_{n,m}} e^{-\delta_{n,m}k}} \sim e^{-\mu_{n,m}} \sum_{k=1}^{n} Ck^{r+1} e^{-\delta_{n,m}k} \sim C\Gamma(r+2) e^{-\mu_{n,m}} \delta_{n,m}^{-r-2}. \tag{42}$$

We therefore have

$$\begin{aligned} \mathrm{Var}(Z_n)\mathrm{Var}(Y_n) - \mathrm{Cov}(Z_n,Y_n)^2 &\sim C^2(\Gamma(r+2)\Gamma(r) - \Gamma(r+1)^2) e^{-2\mu_{n,m}} \delta_{n,m}^{-2r-2} \\ &= C^2 r\Gamma(r) e^{-2\mu_{n,m}} \delta_{n,m}^{-2r-2}. \end{aligned}$$

and

$$I_1 \sim \frac{e^{\mu_{n,m}} \delta_{n,m}^{r+1}}{C\sqrt{r\Gamma(r)}} \int\int_{S_n} e^{-2\pi^2(u^2+v^2)} du\, dv.$$

If we show that $\liminf_{n\to\infty} S_n = \mathbb{R}^2$, then

$$\int\int_{S_n} e^{-2\pi^2(u^2+v^2)} du\, dv \sim \int\int_{\mathbb{R}^2} e^{-2\pi^2(u^2+v^2)} du\, dv = (2\pi)^{-1}$$

will be an immediate consequence and

$$I_1 \sim \frac{e^{\mu_{n,m}} \delta_{n,m}^{r+1}}{2\pi C\sqrt{r\Gamma(r)}} \tag{43}$$

will have been shown. Let ∂R_n and ∂S_n denote the boudaries of R_n and S_n. In view of the identity

$$\inf\left\{ u^2 + v^2 : \begin{pmatrix} u \\ v \end{pmatrix}^T \in \partial S_n \right\} = \inf\left\{ \left| \sqrt{\Sigma_n} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \right|_2^2 : \begin{pmatrix} \alpha \\ \beta \end{pmatrix}^T \in \partial R_n \right\}, \tag{44}$$

where $|\cdot|_2$ represents L_2 distance, and the fact that $(0,0) \in S_n$, if we show that the right hand side of (44) converges to ∞ as $n \to \infty$, then $\liminf_{n\to\infty} S_n = \mathbb{R}^2$ will follow. Observe that

$$\begin{aligned} \inf_{-\beta_0 \le \beta \le \beta_0} \left\{ \left| \sqrt{\Sigma_n} \begin{pmatrix} \alpha_0 \\ \beta \end{pmatrix} \right|_2^2 \right\} &= \inf_{-\beta_0 \le \beta \le \beta_0} \left\{ \mathrm{Var}(Y_n)\alpha_0^2 + 2\mathrm{Cov}(Y_n,Z_n)\alpha_0\beta + \mathrm{Var}(Z_n)\beta^2 \right\} \\ &\ge \inf_{\beta\in\mathbb{R}} \left\{ \mathrm{Var}(Y_n)\alpha_0^2 + 2\mathrm{Cov}(Y_n,Z_n)\alpha_0\beta + \mathrm{Var}(Z_n)\beta^2 \right\}. \end{aligned}$$

The last infimum occurs when

$$\beta = -\frac{\alpha_0 \mathrm{Cov}(Y_n, Z_n)}{\mathrm{Var}(Z_n)}$$

and so

$$\inf_{-\beta_0\le\beta\le\beta_0}\left\{\left|\sqrt{\Sigma_n}\begin{pmatrix}\alpha_0\\ \beta\end{pmatrix}\right|_2^2\right\} \ge \mathrm{Var}(Y_n)\alpha_0^2 - \frac{\alpha_0^2\mathrm{Cov}(Y_n,Z_n)^2}{\mathrm{Var}(Z_n)}$$
$$= \mathrm{Var}(Y_n)\alpha_0^2\left(1-\frac{\mathrm{Cov}(Y_n,Z_n)^2}{\mathrm{Var}(Y_n)\mathrm{Var}(Z_n)}\right)$$

Similarly,

$$\inf_{-\beta_0\le\beta\le\beta_0}\left\{\left|\sqrt{\Sigma_n}\begin{pmatrix}-\alpha_0\\ \beta\end{pmatrix}\right|_2^2\right\} \ge \mathrm{Var}(Y_n)\alpha_0^2\left(1-\frac{\mathrm{Cov}(Y_n,Z_n)^2}{\mathrm{Var}(Y_n)\mathrm{Var}(Z_n)}\right)$$
$$\inf_{-\alpha_0\le\alpha\le\alpha_0}\left\{\left|\sqrt{\Sigma_n}\begin{pmatrix}\alpha\\ \beta_0\end{pmatrix}\right|_2^2\right\} \ge \mathrm{Var}(Z_n)\beta_0^2\left(1-\frac{\mathrm{Cov}(Z_n,Y_n)^2}{\mathrm{Var}(Z_n)\mathrm{Var}(Y_n)}\right)$$
$$\inf_{-\alpha_0\le\alpha\le\alpha_0}\left\{\left|\sqrt{\Sigma_n}\begin{pmatrix}\alpha\\ -\beta_0\end{pmatrix}\right|_2^2\right\} \ge \mathrm{Var}(Z_n)\beta_0^2\left(1-\frac{\mathrm{Cov}(Z_n,Y_n)^2}{\mathrm{Var}(Z_n)\mathrm{Var}(Y_n)}\right)$$

We check using (27), (28), (40), (41), (42) that

$$\mathrm{Var}(Y_n)\alpha_0^2 \sim (C\Gamma(r)e^{-\mu_{n,m}}\delta_{n,m}^{-r})(e^{\mu_{n,m}/2}\delta_{n,m}^{r/2}\log^{(4+\epsilon)/8}n)^2 = C\Gamma(r)\log^{(4+\epsilon)/4}n,$$
$$\mathrm{Var}(Z_n)\beta_0^2 \sim (C\Gamma(r+2)e^{-\mu_{n,m}}\delta_{n,m}^{-r-2})(e^{\mu_{n,m}/2}\delta_{n,m}^{r/2+1}\log^{(8+\epsilon)/16}n)^2 = C\Gamma(r+2)\log^{(8+\epsilon)/8}n,$$

and

$$1-\frac{\mathrm{Cov}(Z_n,Y_n)^2}{\mathrm{Var}(Z_n)\mathrm{Var}(Y_n)} \sim 1-\frac{\Gamma(r+1)^2}{\Gamma(r)\Gamma(r+2)} > 0, \tag{45}$$

which implies

$$\lim_{n\to\infty}\inf\left\{\left|\sqrt{\Sigma_n}\begin{pmatrix}\alpha\\ \beta\end{pmatrix}\right|_2^2 : \begin{pmatrix}\alpha\\ \beta\end{pmatrix}^T\in\partial R_n\right\}=\infty.$$

■

Estimate of I_2

Similarly to a calculation in the proof of Lemma 3 of [7], we have

$$\begin{aligned}\log|\phi_n(\alpha,\beta)| &= \Re(\log(\phi_n(\alpha,\beta))\\ &= \Re\left(-\sum_{k=1}^n b_k\log\left(\frac{1-e^{-\mu_{n,m}+2\pi i\alpha-\delta_{n,m}k+2\pi i\beta k}}{1-e^{-\mu_{n,m}-\delta_{n,m}k}}\right)\right)\\ &= -\frac{1}{2}\sum_{k=1}^n b_k\log\left(1+\frac{4e^{-\mu_{n,m}-\delta_{n,m}k}\sin^2(\pi\alpha+\pi\beta k)}{(1-e^{-\mu_{n,m}-\delta_{n,m}k})^2}\right)\\ &\le -\frac{1}{2}\sum_{k=1}^n b_k\log\left(1+4e^{-\mu_{n,m}-\delta_{n,m}k}\sin^2(\pi\alpha+\pi\beta k)\right)\\ &\le -\frac{\log 5}{2}\sum_{k=1}^n b_k e^{-\mu_{n,m}-\delta_{n,m}k}\sin^2(\pi\alpha+\pi\beta k)\end{aligned}$$

and the application of (3.70) in [6] gives

$$\log|\phi_n(\alpha,\beta)| \le -2\log 5\sum_{k=1}^n b_k e^{-\mu_{n,m}-\delta_{n,m}k}\parallel\alpha+\beta k\parallel^2 \tag{46}$$

where $\{x\}$ is defined to be the fractional part of x and

$$\parallel x\parallel = \begin{cases}\{x\} & \text{if } \{x\}\le 1/2;\\ 1-\{x\} & \text{if } \{x\}>1/2.\end{cases}$$

Define

$$V_n(\alpha,\beta) = \sum_{k=1}^{n} b_k e^{-\mu_{n,m}-\delta_{n,m}k} \parallel \alpha+\beta k \parallel^2 .$$

We will find lower bounds for $V_n(\alpha,\beta)$ on four regions which partiton $\overline{R_n}$.

First, suppose that $\alpha_0 < |\alpha| \le 1/2$ and $|\beta| \le \beta_0$. Note that for such β,

$$\begin{aligned}|\beta\delta_{n,m}^{-1}| &\le \beta_0\delta_{n,m}^{-1}\\ &= e^{\mu_{n,m}/2}\delta_{n,m}^{\rho_r/2}\log^{(8+\epsilon)/16} n\\ &= o(\alpha_0(n)).\end{aligned}$$

By the definition of $\parallel x \parallel$ we have

$$\parallel x+y \parallel \ge \parallel x \parallel - |y| \quad \forall x,y\in\mathbb{R}.$$

Therefore, for all $1 \le k \le \delta_{n,m}^{-1}$,

$$\begin{aligned}\parallel \alpha+\beta k \parallel &\ge \parallel \alpha \parallel - |\beta\delta_{n,m}^{-1}|\\ &\ge \alpha_0 - |\beta\delta_{n,m}^{-1}|\\ &= \alpha_0(1+o(1)).\end{aligned}$$

It follows that

$$\begin{aligned}V_n(\alpha,\beta) &\ge (1+o(1))e^{-\mu_{n,m}}\alpha_0^2\sum_{k=1}^{\delta_{n,m}^{-1}} b_k e^{-\delta_{n,m}k}\\ &\sim Ce^{-\mu_{n,m}}\alpha_0^2\sum_{k=1}^{\delta_{n,m}^{-1}} k^{r-1}e^{-\delta_{n,m}k}\\ &\asymp e^{-\mu_{n,m}}\alpha_0^2\delta_{n,m}^{-r}\\ &= \log^{(4+\epsilon)/4} n. \end{aligned} \tag{47}$$

Suppose that $|\alpha| \le 1/4$ and $\beta_0 < |\beta| \le \delta_{n,m}$. Define $F_i(u) = \int_0^u v^{r-1+i}e^{-v}\,dv$, $x \ge 0$, $i = 0,1,2$. Observe that $F_i(u) = \frac{u^{r+i}}{r+i} + O(u^{r+i+1})$, $u \to 0$, and that therefore

$$F_1(u)^2 - F_0(u)F_2(u) = -\frac{1}{r(r+2)(r+1)^2}u^{2r+2} + O(u^{2r+3}).$$

Choose $0 < u_0 < 1/4$ small enough so that $F_1(u_0)^2 - F_0(u_0)F_2(u_0) < 0$. Then for all $0 \le k \le u_0\delta_{n,m}^{-1}$, we have $|\alpha+\beta k| \le |\alpha| + |\beta k| \le 1/4 + 1/4 = 1/2$. Therefore, for all such k, $\parallel \alpha+\beta k \parallel = |\alpha+\beta k|$ and

$$\begin{aligned}V_n(\alpha,\beta) &\ge \sum_{k=1}^{u_0\delta_{n,m}^{-1}} b_k e^{-\mu_{n,m}-\delta_{n,m}k}(\alpha+\beta k)^2\\ &\sim Ce^{-\mu_{n,m}}\left(\alpha^2\sum_{k=1}^{u_0\delta_{n,m}^{-1}} k^{r-1}e^{-\delta_{n,m}k} + 2\alpha\beta\sum_{k=1}^{u_0\delta_{n,m}^{-1}} k^{r}e^{-\delta_{n,m}k} + \beta^2\sum_{k=1}^{u_0\delta_{n,m}^{-1}} k^{r+1}e^{-\delta_{n,m}k}\right)\\ &\sim Ce^{-\mu_{n,m}}\left(\alpha^2\delta_{n,m}^{-r}F_0(u_0) + 2\alpha\beta\delta_{n,m}^{-r-1}F_1(u_0) + \beta^2\delta_{n,m}^{-r-2}F_2(u_0)\right)\\ &= Ce^{-\mu_{n,m}}\delta_{n,m}^{-r-2}\beta^2\left(F_0(u_0)x^2 + 2F_1(u_0)x + F_3(u_0)\right),\end{aligned}$$

where $x = \alpha\beta^{-1}\delta_{n,m}$. Because the quadratic $F_0(u_0)x^2+2F_1(u_0)x+F_3(u_0)$ has discriminant $4(F_1(u_0)^2-F_0(u_0)F_2(u_0)) < 0$ and $F_0(u_0) > 0$, there is a constant $K > 0$ such that $f(x) > K$ for all $x \in \mathbb{R}$. Therefore, for n large enough we have

$$V_n(\alpha,\beta) > CKe^{-\mu_{n,m}}\delta_{n,m}^{-r-2}\beta_0^2 = CK\log^{(8+\epsilon)/8} n. \tag{48}$$

Suppose that $1/4 < |\alpha| \le 1/2$ and $\beta_0 < |\beta| \le \delta_{n,m}$. Then, for $1 \le k \le \delta_{n,m}^{-1}/8$,

$$\| \alpha + \beta k \| \ge \| \alpha \| - |\beta k| \ge 1/4 - \delta_{n,m}(\delta_{n,m}^{-1}/8) = 1/8$$

and

$$\begin{aligned} V_n(\alpha,\beta) &\ge \frac{1}{64} \sum_{k=1}^{\delta_{n,m}^{-1}/8} b_k e^{-\mu_{n,m}-\delta_{n,m}k} \\ &\sim \frac{Ce^{-\mu_{n,m}}}{64} \sum_{k=1}^{\delta_{n,m}^{-1}/8} k^{r-1} e^{-\delta_{n,m}k} \\ &\asymp e^{-\mu_{n,m}} \delta_{n,m}^{-r} \\ &\asymp (m^{r+1}n^{-r})(m^{-r}n^{r}) \\ &= m \\ &\ge \log^{3+\epsilon} n \end{aligned} \tag{49}$$

for n large enough.

Finally, suppose that $|\alpha| \le 1/2$ and $\delta_{n,m}^{-1} < |\beta| \le 1/2$. Define

$$Q(\alpha,\beta;n) = \{1 \le k \le n : \| \alpha + \beta k \| \ge 1/4\}.$$

Clearly,

$$\begin{aligned} Q(\alpha,\beta;n) &= \{1 \le k \le n : j + 1/4 \le \alpha + \beta k \le j + 3/4, \quad j = 0,1,\ldots\} \\ &= \{1 \le k \le n : j + (1/4 - \alpha) \le \beta k \le j + (3/4 - \alpha), \quad j = 0,1,\ldots\}. \end{aligned}$$

Routine modifications to the estimates on pages 18 and 19 of [6] which produce the lower bound (3.77) in that paper result in

$$\sum_{k=1}^{n} b_k e^{-\delta_{n,m}k} \| \alpha + \beta k \|^2 \ge \eta \delta_{n,m}^{-\rho_r}.$$

for a constant $\eta > 0$ and therefore

$$V_n(\alpha,\beta) \ge \eta e^{-\mu_{n,m}} \delta_{n,m}^{-r} \asymp m > \log^{3+\epsilon} n \tag{50}$$

for n large enough.

Combining (47), (48), (49) and (50) shows that $V(\alpha,\beta) \ge \Theta(\log^{(8+\epsilon)/8} n)$ uniformly for $(\alpha,\beta) \in \overline{R_n}$. From this lower bound and (46) it follows that

$$I_2 = O\left(\exp(-\Theta(\log^{(8+\epsilon)/8} n)\right). \tag{51}$$

Note that (10), (11) and (43) imply

$$I_1 \asymp (m^{r+1}n^{-r})^{-1}(mn^{-1})^{r+1} = n^{-1}. \tag{52}$$

Together, (29), (43), (51) and (52) imply (26). ∎

Proof of Theorem 1

The theorem follows from (22), Lemma 3, and Lemma 4.

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